International Journal of Theoretical Physics, Vol. 30, No. 11, 1991

Fermi Coordinate Transformation under Baseline Change in Relativistic Celestial Mechanics

J. M. Gambi,[1] P. Romero,[1] A. San Miguel,[2] and F. Vicente[2]

Received October 16, 1990

A relativistic description for the transformation under a change of baseline between the Fermi coordinates associated to two nonrotating local reference frames is carried out by using techniques based on the world function and the parallel propagator to space-times of small curvature. By assuming one of the observers in free motion, the transformation is evaluated in the first order of approximation with respect to the Riemann tensor and under the assumption of quasiparallelism between the two baselines.

1. INTRODUCTION

Since a precise interpretation of astrometric data ultimately depends upon the validity of the reference frames constructed in relativistic celestial mechanics, seeking appropriate reference frames and comprehensive descriptions for the connection between the coordinates naturally associated to them has become an important issue to derive that interpretation in a proper way, as required by modern techniques such as SLR, LLR, and VLBI.

After several proposals made during recent years on the basis of Synge's construction, some of them to avoid practical problems derived from the post-Newtonian dynamic formalism commonly used to describe the solar system (see, for example, Fukushima *et al.*, 1986), it is becoming apparent that the proper reference frame constructed by Synge (1960), as well as the one provided by Misner *et al.* (1973), remain valid not only due to their clear and easy interpretation, but because, being the most natural choice in general relativity, if a further analysis of the inconveniences arising from their choice is taken into account, they also appear as still probably the most natural extensions of the classical nonrotating and rotating reference frames,

[1]Instituto de Astronomia y Geodesia (UCM-CSIC), Facultad de Ciencias Matematicas, Universidad Complutense, Madrid, Spain.
[2]Departamento de Matematica Aplicada, Universidad de Valladolid, Valladolid, Spain.

respectively, whatever the origin, topocentric, terrestrial, or solar system barycentric, considered.

In any case, since the main conceptual contributions come from the construction of these two reference frames and, on the other hand, the meaning of their naturally associated coordinates, both polar and Fermi, is already well known, in this paper attention will be paid to the second part of the problem, i.e., to the relationship between the (Fermi) coordinates associated to two of these frames.

To do this we take Brumberg's (1979) suggestion and, therefore, the techniques based on the world function and the parallel propagator are systematically used following Synge's method (1960). On the other hand, since any proper reference frame, as constructed by Misner *et al.*, can be obtained by a three-dimensional rotation from a nonrotating proper reference frame attached to the same observer's baseline as constructed by Synge, the analysis is mainly focused on the connection between reference frames of this last kind under a change of baseline.

Furthermore, to keep the situation under control some realistic assumptions are successively made: First, since the problem reduces to a geometric transformation based on properties of geodesic triangles with at least one non-null side, one of the observer's baselines is supposed to be a (timelike) geodesic of the space-time, but the other baseline is not restricted in any way. This assumption is realistic because, in this case, one of the observers will be in free motion and the other, as it happens with a topocentric observer, will not. Next, after obtaining a general relationship, the space-time itself is supposed to be of small curvature to resemble the solar system and the calculations are carried out in a first order of approximation with respect to the Riemann tensor. As a matter of consistency, the resulting formulas are also interpretated in the flat-space-time particular case. Finally, the last transformation is derived by assuming that the baselines of the two observers are quasiparallel, which is an assumption really valid in any physical situation.

2. THE WORLD FUNCTION AND THE PARALLEL PROPAGATOR

We are, therefore, in the pseudo-Riemann space-time $(\mathcal{M}, g)$ of general relativity and a region $\mathcal{N}$ is considered in it so that any two points P and Q in this region determine a unique geodesic Γ_{PQ} passing through them. Γ_{PQ} is supposed to be given, in local coordinates x^i, by $x^i = x^i(u)$, where $u \in [u_0, u_1]$ is a special affine parameter in the sense that the equations for Γ_{PQ} read $\delta U^i/\delta u = 0$, where $U^i = dx^i/du$ (Latin indices range from 1 to 4 and Greek from 1 to 3).

As was said, to study the problem under consideration, techniques based on the world function are used. A detailed discussion of these techniques can be found in Synge (1960). Here we only give a brief summary. In the region $\mathcal{N}$ this function $\Omega(PQ)$ is defined by the integral

$$\Omega(PQ) = \Omega(x^{i_P}, x^{j_Q}) = \tfrac{1}{2}(u_1 - u_0) \int_{u_0}^{u_1} g_{ij} U^i U^j \, du \tag{2.1}$$

taken along Γ_{PQ} with $x^{i_P} = x^i(u_0)$ and $x^{i_Q} = x^i(u_1)$. This integral, which is a function of the eight coordinates of P and Q, has a value independent of the particular parameter chosen; it is, in fact, half the square of the measure of the geodesic joining $P(x^{i_P})$ and $Q(x^{i_Q})$; and, therefore, covariant differentiation can be carried out with respect to the coordinates of P or with respect to those of Q. [To avoid cumbersome notation, these covariant derivatives are denoted by simple subscripts without the usual vertical stroke and the operations will be applied in the order of the subscripts. Thus, the covariant derivatives of $\Omega(PQ)$ with respect to the coordinates of P and those of Q are denoted by $\Omega_{i_P}(PQ)$ and $\Omega_{i_Q}(PQ)$, respectively.] Of course, both $\Omega_{i_P}(PQ)$ and $\Omega_{i_Q}(PQ)$ are again functions of the eight coordinates of P and Q. They are two-point tensors. For example, $\Omega_{i_P}(PQ)$ is a covariant vector under transformations at P, and an invariant under transformations at Q. In fact, it can be shown that

$$\Omega_{i_P}(PQ) = -(u_1 - u_0) U_{i_P}, \qquad \Omega_{i_Q}(PQ) = (u_1 - u_0) U_{i_Q} \tag{2.2}$$

U^{i_P} and U^{i_Q} being the tangent vectors to Γ_{PQ} at P and Q, respectively, and

$$U_{i_P} = g_{i_P j_P} U^{j_P}, \qquad U_{i_Q} = g_{i_Q j_Q} U^{j_Q} \tag{2.3}$$

where $g_{i_P j_P}$ and $g_{i_Q j_Q}$ are the metric tensor g_{ij} at P and Q, respectively. Therefore, Ω^{i_P} belongs to the tangent spaces of $\mathcal{M}$ at P, $T_P\mathcal{M}$, and Ω^{i_Q} belongs to $T_Q\mathcal{M}$ (in general, v^{i_P} denotes the components of the vector $\mathbf{v} \in T_P\mathcal{M}$ with respect to the coordinate basis of $T_P\mathcal{M}$).

If the case where Γ_{PQ} is not null were considered, we would have

$$\Omega_{i_P}(PQ) = -L\lambda_{i_P}, \qquad \Omega_{i_Q}(PQ) = L\lambda_{i_Q} \tag{2.4}$$

where λ^{i_P} and λ^{i_Q} are the unit tangent vectors of Γ_{PQ} at P and Q, respectively, and for their norms we would have

$$\begin{aligned} g_{i_P j_P} \Omega^{i_P}(PQ) \Omega^{j_P}(PQ) &= 2\Omega(PQ) = L^2 \\ g_{i_Q j_Q} \Omega^{i_Q}(PQ) \Omega^{j_Q}(PQ) &= 2\Omega(PQ) = L^2 \end{aligned} \tag{2.5}$$

so that $-\Omega^{i_P}(PQ)$ and $\Omega^{i_Q}(PQ)$ appear as tangent vectors at P and Q, respectively, with magnitude equal to the measure L of the geodesic Γ_{PQ}. [In this regard, it is perhaps more suggestive to think of a standard notation like $\overline{PQ}$ instead of $-\Omega^{i_P}(PQ)$, and $\overline{QP}$ instead of $-\Omega^{i_Q}(PQ)$, and to consider these vectors as "position vectors" in a similar way as we properly do in the flat-space-time case.]

Now, if for a given point P the components of any vector $\mathbf{v} \in T_P\mathcal{M}$ with respect to an orthonormal basis $\lambda^{i_P}_{(a)}$ are denoted by $v^{(a)}$ $(a = 1, 2, 3, 4)$ (we shall label the vectors so that $\lambda^{i_P}_{(4)}$ is timelike), then the conditions of orthonormality may be written

$$\lambda^{i_P}_{(a)}\lambda_{(b)i_P} = \eta_{(ab)} \tag{2.6}$$

where $\eta_{(ab)} = \eta^{(ab)} = \mathrm{diag}(1, 1, 1, -1)$, so that, although the labels (a) have no tensorial meaning, we shall raise and lower them by means of the η-matrix. Thus, we have

$$\lambda^{(a)i_P} = \eta^{(ab)}\lambda^{i_P}_{(b)}, \qquad \lambda^{(a)}_{i_P} = \eta^{(ab)}\lambda_{(b)i_P} \tag{2.7}$$

so that if two orthonormal tetrads $\lambda^{i_P}_{(a)}$, $\mu^{i_P}_{(b)}$ are attached to the same point P, then they are related by the Lorentz transformation

$$L^{(a)}_{(b)} = \lambda^{(a)}_{i_P}\mu^{i_P}_{(b)} \tag{2.8}$$

so that

$$\lambda^{(a)}_{j_P} = L^{(a)}_{(b)}\mu^{(b)}_{j_P}, \qquad \mu^{j_P}_{(b)} = L^{(a)}_{(b)}\lambda^{j_P}_{(a)} \tag{2.9}$$

On the other hand, if P, Q are two points in $\mathcal{N}$, and $\lambda^{i_P}_{(a)}$ is an orthonormal basis of $T_P\mathcal{M}$ carried by parallel transport along the geodesic Γ_{PQ}, we can obtain another orthonormal basis $\lambda^{i_Q}_{(a)}$ at Q by means of the parallel propagator $g_{i_Pj_Q}$, which is a two-point symmetric tensor defined by

$$g_{i_Pj_Q} = \lambda_{(a)i_P}\lambda^{(a)}_{j_Q} = \eta_{(ab)}\lambda^{(a)}_{i_P}\lambda^{(b)}_{j_Q} \tag{2.10}$$

so that

$$g^{i_P}_{j_Q} = g^{i_PK_P}g_{K_Pj_Q}, \qquad g^{i_Pj_Q} = g^{i_PK_P}g_{K_Pm_Q}g^{m_Qj_Q} \tag{2.11}$$

and

$$\lambda_{(a)j_Q} = g_{i_Pj_Q}\lambda^{i_P}_{(a)}, \qquad \lambda_{(a)i_P} = g_{i_Pj_Q}\lambda^{j_Q}_{(a)} \tag{2.12}$$

Then, by using the solution for finite geodesic triangles, the mentioned general expression for the transformation of Fermi coordinates is obtained in the next section [a detailed discussion of this geodesic solution can also be found in Synge (1960)].

3. GENERAL TRANSFORMATION

Let us suppose that Π is a timelike curve in $\mathcal{N}$ that is parametrized with the affine parameter s, so that A represents a generic point in it and let us denote by n^{i_A} the unit tangent vector of Π at A, and by $\Sigma(A)$ the hypersurface of $\mathcal{N}$ defined by

$$\Sigma(A) := \{P \in \mathcal{N} / n_{i_A} \Omega^{i_A}(AP) = 0\} \tag{3.1}$$

This hypersurface is made out of all the geodesics corresponding to the Cauchy data $(P, \Omega^{i_A}(AP))$ that intersect orthogonally to Π at A. If, in the same way, for a timelike geodesic Γ in $\mathcal{N}$, without common points with Π and with affine parameter s' and current point D, we consider the unit tangent vector n^{i_D}, and

$$\Sigma(D) := \{Q \in \mathcal{N} / n_{i_D} \Omega^{i_D}(DQ) = 0\} \tag{3.2}$$

we can define a field of orthonormal tetrads $\mathscr{C}$ on $\Sigma(A)$ by parallel transporting an orthonormal basis $\lambda^{i_A}_{(a)}$ of $T_A\mathscr{M}$ (chosen so that $\lambda^{i_A}_{(4)} = n^{i_A}$) to any point $P \in \Sigma(A)$ along the geodesic Γ_{AP}, i.e.,

$$\lambda^{i_P}_{(a)} = g^{i_P}_{j_A} \lambda^{j_A}_{(a)} \tag{3.3}$$

and, similarly, we will have the field $\mathscr{C}'$ of tetrads $\lambda^{i_Q}_{(a)}$ on $\Sigma(D)$ by parallel transporting an orthonormal basis $\lambda^{i_D}_{(a)}$ of $T_D\mathscr{M}$, with $\lambda^{i_D}_{(4)} = n^{i_D}$, so that

$$\lambda^{i_Q}_{(a)} = g^{i_Q}_{j_D} \lambda^{j_D}_{(a)} \tag{3.4}$$

Then the Fermi coordinates of a point $B \in \Sigma(A) \cap \Sigma(D)$ can be written in terms of the world function in the following way: First, with respect to the baseline Π and to the tetrad $\lambda^i_{(a)}$ Fermi transported along Π, which we shall denote by $x^{(\alpha)}(B) = x_{(\alpha)}(B)$, $x^{(4)}(B) = -x_{(4)}(B) = s$, as

$$X^{(\alpha)}(B)|_A = X_{(\alpha)}(B)|_A = -\Omega_{i_A}(AB)\lambda^{i_A}_{(\alpha)}, \qquad X^{(4)}(B)|_A = -X_{(4)}(B)|_A = s \tag{3.5}$$

where $\alpha = 1, 2, 3$, so that, according to the interpretation given previously, by specifying the point A, they can be thought of as components of the "position vector" of B with respect to the origin A and the basis $\lambda^{i_A}_{(a)}$; and second, with respect to the geodesic line Γ and the parallel-transported tetrad $\lambda'^i_{(a)}$ along Γ, which will be denoted by

$$x'^{(\alpha)}(B) = x'_{(\alpha)}(B), \; x'^{(4)}(B) = -x'_{(4)}(B) = s'$$

as

$$X^{(\alpha)}(B)|_D = X_{(\alpha)}(B)|_D = -\Omega_{i_D}(DB)\lambda^{i_D}_{(\alpha)}, \qquad X^{(4)}(B)|_D = -X_{(4)}(B)|_D = s' \tag{3.6}$$

Now, in order to obtain the relationship between the Fermi coordinates $[x_{(\alpha)}(B), x_{(4)}(B)]$ and $[x'_{(\alpha)}(B), x'_{(4)}(B)]$, let us suppose that C is the point of intersection of $\Sigma(A)$ and Γ, and that $X_{(4)}(C)|_D = -\Delta s'$. Then, considering the geodesic quadrilateral *ABCD* and the geodesic diagonal Γ_{AD}, we have for the triangle *ADB*

$$\Omega(AD)=\Omega(AB)+\Omega(BD)-\Omega_{i_B}(BA)\Omega^{i_B}(BD)+\tfrac{1}{2}\phi(ABD) \tag{3.7}$$

where $\phi(ABD)$ is a three-point invariant given by

$$\phi(ABD)=\tfrac{1}{3}\int_0^1 (1-\zeta)^3\,\frac{d^4\Omega(\zeta)}{d\zeta^4}\,d\zeta \tag{3.8}$$

provided that Γ_{AB} and Γ_{BD} are parametrized so that ζ takes the values 0 at *B*, and 1 both at *A* and *D*. Similarly, for the triangle *ACD* we have

$$\Omega(AD)=\Omega(AC)+\Omega(CD)-\Omega_{i_C}(AC)\Omega^{i_C}(CD)+\tfrac{1}{2}\phi(ACD) \tag{3.9}$$

where

$$\phi(ACD)=\tfrac{1}{3}\int_0^1 (1-\theta)^3\,\frac{d^4\Omega(\theta)}{d\theta^4}\,d\theta \tag{3.10}$$

and θ is an affine parameter on Γ_{CA} and Γ_{CD} satisfying $\theta=0$ at *C*, and $\theta=$ 1 both at *A* and *D*.

Now, taking the covariant derivatives of (3.7) and (3.9) with respect to the coordinates of *A*, we have

$$\Omega_{i_A}(BA)-\chi_{i_A}(ABD)+\tfrac{1}{2}\phi_{i_A}(ABD)=\Omega_{i_A}(CA)-\chi_{i_A}(ACD)+\tfrac{1}{2}\phi_{i_A}(ACD) \tag{3.11}$$

and taking covariant derivatives of (3.7) and (3.9) with respect to the coordinates of *D*, we have

$$\Omega_{i_D}(BD)-\chi_{i_D}(ABD)+\tfrac{1}{2}\phi_{i_D}(ABD)=\Omega_{i_D}(CD)-\chi_{i_D}(ACD)+\tfrac{1}{2}\phi_{i_D}(ACD) \tag{3.12}$$

where $\chi(ABD)$ and $\chi(ACD)$ are the three-point invariants

$$\chi(ABD)=\Omega_{i_B}(AB)\Omega^{i_B}(BD), \qquad \chi(ACD)=\Omega_{i_C}(AC)\Omega^{i_C}(CD) \tag{3.13}$$

Now, by parallel transporting the basis $\lambda^{i_A}_{(a)}$ at *A* to the point *D* along the path $\Gamma_{AC}-\Gamma_{CD}$, we have a new basis $\mu^{i_D}_{(a)}$ at *D* that, passing through a basis at *C*, $\mu^{i_C}_{(a)}$, is given by

$$\mu^{i_D}_{(a)}=g^{i_D}_{j_C}g^{j_C}_{k_A}\lambda^{k_A}_{(a)}=g^{i_D}_{j_C}\mu^{j_C}_{(a)} \tag{3.14}$$

so that the old basis $\lambda'^{i_D}_{(a)}$ and this last one $\mu^{i_D}_{(a)}$ are related according to (2.9) by

$$\mu^{i_D}_{(a)}=L^{(b)}_{(a)}\lambda'^{i_D}_{(b)} \tag{3.15}$$

where $L^{(b)}_{(a)}$ is a general Lorentz transformation (boost plus spatial rotation). On the other hand, (3.11) can be transformed into another expression by applying successively the two parallel propagators $g^{i_A}_{j_C}$ and $g^{j_C}_{k_D}$ to it,

$$g^{i_A}_{j_C} g^{j_C}_{k_D}[\Omega_{i_A}(BA) - \chi_{i_A}(ABD) + \tfrac{1}{2}\phi_{i_A}(ABD)]$$
$$= g^{i_A}_{j_D} g^{j_C}_{k_D}[\Omega_{i_A}(CA) - \chi_{i_A}(ACD) + \tfrac{1}{2}\phi_{i_A}(ACD)] \qquad (3.16)$$

so that, if the inner product of this expression by $\mu^{i_D}_{(a)}$ is evaluated, we have

$$\Omega_{i_A}(BA)\lambda^{i_A}_{(a)} - \chi_{i_A}(ABD)\lambda^{i_A}_{(a)} + \tfrac{1}{2}\phi_{i_A}(ABD)\lambda^{i_A}_{(a)}$$
$$= \Omega_{i_A}(CA)\lambda^{i_A}_{(a)} - \chi_{i_A}(ACD)\lambda^{i_A}_{(a)} + \tfrac{1}{2}\phi_{i_A}(ACD)\lambda^{i_A}_{(a)} \qquad (3.17)$$

On the other hand, taking into account (3.15), the inner product of (3.12) by $\mu^{i_D}_{(a)}$ gives

$$[\Omega_{i_D}(BD) - \chi_{i_D}(ABD) + \tfrac{1}{2}\phi_{i_D}(ABD)]L^{(b)}_{(a)}\lambda'^{i_D}_{(b)}$$
$$= [\Omega_{i_D}(CD) - \chi_{i_D}(ACD) + \tfrac{1}{2}\phi_{i_D}(ACD)]L^{(b)}_{(a)}\lambda'^{i_D}_{(b)} \qquad (3.18)$$

so that, since $\Omega_{i_D}(BD)\lambda'^{i_D}_{(4)} = 0$ and $\Omega_{i_D}(CD)\lambda'^{i_D}_{(a)} = 0$, we have

$$\Omega_{i_D}(BD)L^{(\beta)}_{(a)}\lambda'^{i_D}_{(\beta)} + [-\chi_{i_D}(ABD) + \tfrac{1}{2}\phi_{i_D}(ABD)]L^{(b)}_{(a)}\lambda'^{i_D}_{(b)}$$
$$= \Omega_{i_D}(CD)L^{(4)}_{(a)}\lambda'^{i_D}_{(4)} + [-\chi_{i_D}(ACD) + \tfrac{1}{2}\phi_{i_D}(ACD)]L^{(b)}_{(a)}\lambda'^{i_D}_{(b)} \qquad (3.19)$$

and therefore, from (3.5), (3.6), (3.17), and (3.19), we finally have

$$X_{(a)}(B)|_A - X_{(a)}(C)|_A - X_{(\beta)}(B)|_D L^{(\beta)}_{(a)}$$
$$= \Omega_{i_D}(CD)L^{(4)}_{(a)}\lambda'^{i_D}_{(4)} - [\chi_{i_A}(ABD) - \chi_{i_A}(ACD)]\lambda^{i_A}_{(a)}$$
$$+ [\chi_{i_D}(ABD) - \chi_{i_D}(ACD)]L^{(b)}_{(a)}\lambda'^{i_D}_{(b)} + \tfrac{1}{2}[\phi_{i_A}(ABD) - \phi_{i_A}(ACD)]\lambda^{i_A}_{(a)}$$
$$- \tfrac{1}{2}[\phi_{i_D}(ABD) - \phi_{i_D}(ACD)]L^{(b)}_{(a)}\lambda'^{i_D}_{(b)} \qquad (3.20)$$

This is the general relationship sought. Thinking of Fermi coordinates as components of position vectors in the way previously said, we can see in (3.20) (as well as in the procedure to get it) a generalization of the flat-space-time case. This case is obviously much simpler since, as we will see, we will have instead of (3.20) the following relationship:

$$X_{(a)}(B)|_A - X_{(a)}(C)|_A - X_{(\beta)}(B)|_D L^{(\beta)}_{(a)} = 0 \qquad (3.21)$$

Therefore, on the basis of this interpretation it can be said that, except for particular cases, the Riemann tensor manifest itself in the general need of having to relate four points A, B, C, and D instead of only three A, B, and D (to account for the nonsynchronism between the two baselines, through C

and D) and in the general impossibility of drawing polygonals with "touching tips" as in the flat-space-time case. In fact, if this were the case, then we would have

$$\Omega_{i_D}(CD)L^{(4)}_{(a)}\lambda'^{i_D}_{(4)}=0 \tag{3.22}$$

because in this case $\Sigma(A)=\Sigma(D)$, so that $C=D$, $\lambda'^{i_D}_{(a)}=\mu^{i_D}_{(a)}$, and, therefore, $L^{(4)}_{(a)}=0$. On the other hand, since in this case the Riemann tensor is null, then the two last terms of the right-hand side of (3.20) are null because the integrands in $\phi(ACD)$ and $\phi(ABD)$ depend on this tensor [see (4.16) and (5.5)], and for the other two terms we would have as equivalent to them the following expression:

$$\begin{aligned}&-[\Omega_{j_Bi_A}(BA)\Omega^{j_B}(BD)-\Omega_{l_Ci_A}(CA)\Omega^{l_C}(CD)]\lambda^{i_A}_{(a)}\\&\quad+[\Omega_{j_B}(BA)\Omega^{j_B}_{k_D}(BD)-\Omega_{l_C}(CA)\Omega^{l_C}_{k_D}(CD)]L^{(b)}_{(a)}\lambda'^{k_D}_{(b)}\end{aligned} \tag{3.23}$$

Now, since in this case $\Omega_{j_Bi_A}(BA)=-g_{j_Bi_A}$, and we also have

$$\begin{aligned}&\Omega^{j_B}(BD)=-g^{j_B}_{k_D}\Omega^{k_D}(BD), \qquad \Omega^{l_C}(CD)=-g^{l_C}_{k_D}\Omega^{k_D}(CD)\\&\Omega_{j_B}(BA)=-g^{i_A}_{j_B}\Omega_{i_A}(BA), \qquad \Omega_{l_C}(CA)=-g^{i_A}_{l_C}\Omega_{i_A}(CA)\end{aligned} \tag{3.24}$$

then (3.23) reduces to

$$\begin{aligned}&-g_{j_Bi_A}g^{j_B}_{k_D}\Omega^{k_D}(BD)\lambda^{i_A}_{(a)}+g_{l_Ci_A}g^{l_C}_{k_D}\Omega^{k_D}(CD)\lambda^{i_A}_{(a)}\\&\quad+g^{i_A}_{j_B}\Omega_{i_A}(BA)g^{j_B}_{k_D}L^{(b)}_{(a)}\lambda'^{k_D}_{(b)}-g^{i_A}_{l_C}\Omega_{i_A}(CA)g^{l_C}_{k_D}L^{(b)}_{(a)}\lambda'^{k_D}_{(b)}\end{aligned} \tag{3.25}$$

or, taking into account that in this case

$$g_{j_Bi_A}g^{j_B}_{K_D}=g_{i_Ak_D}=g_{i_Al_C}g^{l_C}_{k_D} \tag{3.26}$$

so that

$$g_{j_Bi_A}g^{j_B}_{K_D}\lambda^{i_A}_{(a)}=\mu_{(a)k_D}=L^{(b)}_{(a)}\lambda'_{(b)k_D} \tag{3.27}$$

then from (3.25) we would have, as equivalent to (3.23), the expression

$$-\Omega^{k_D}(BD)L^{(b)}_{(a)}\lambda'_{(b)k_D}+\Omega^{k_D}(CD)\mu_{(a)k_D}+\Omega_{i_A}(BA)\lambda^{i_A}_{(a)}-\Omega_{i_A}(CA)\lambda^{i_A}_{(a)} \tag{3.28}$$

or, what is the same, taking into account (3.5) and (3.6), and that $\Omega^{k_D}(CD)\mu_{(a)k_D}=0$,

$$L^{(\beta)}_{(a)}X_{(\beta)}(B)|_D-X_{(\alpha)}(B)|_A+X_{(\alpha)}(C)|_A \tag{3.29}$$

which, together with (3.20), gives the final relationship

$$X_{(\alpha)}(B)|_A-X_{(\alpha)}(D)|_A-X_{(\beta)}(B)|_DL^{(\beta)}_{(a)}=0 \tag{3.30}$$

as expected.

4. APPROXIMATION BY SMALL CURVATURE

Although the relationship (3.20), as general as it is, is not operative in any astrometric situation, two hypotheses are still available to derive a formula that is valid to approach a realistic situation without having to deal with any particular model or predetermined formalism, which is, according to Synge's scheme, the desired procedure. The first simplification can be achieved by assuming the space-time to be of small curvature and this hypothesis is applied to (3.20) in this section. The second one, concerning quasiparallelism, is deferred to the next section.

Given an orthonormal tetrad $\lambda^{i_P}_{(a)}$ of the vector space $T_P\mathscr{M}$ and denoting as before by $v^{(a)}$ the nonholonomic components of a vector $\mathbf{v}\in T_P\mathscr{M}$, a space-time of small curvature is understood in the sense that the norm $\|\cdot\|_P$ on $T_P\mathscr{M}$, given by

$$\|\mathbf{v}\|_P = \max_{a=1,2,3,4} |v^{(a)}| \tag{4.1}$$

extended to any tensor space generated by $T_P\mathscr{M}$ is the one used to introduce the hypothesis that, at any point P of the hypersurfaces $\Sigma(A)$ and $\Sigma(D)$, the Riemann tensor and its covariant derivatives are small, of the first order, or O_1, with respect to the norm (4.1) associated to the fields of orthonormal tetrads $\mathscr{C}$ and $\mathscr{C}'$ defined on $\Sigma(A)$ and $\Sigma(D)$ in the previous section. Then, denoting by O_2 second-order smallness with respect to the Riemann tensor, we start this section by observing that in any geodesic triangle OPQ in $\mathscr{N}$ the following relation holds:

$$g^{j_Q}_{i_P} = g^{l_O}_{i_P} g^{j_Q}_{l_O} - h^{j_Q}_{i_P} + g^{l_O}_{i_P} h^{j_Q}_{l_O} + h^{l_O}_{i_P} g^{j_Q}_{l_O} - \tfrac{1}{2}\phi^{j_Q}_{i_P}(OPQ) + O_2 \tag{4.2}$$

This can be verified by taking into account that in a space-time of small curvature the second covariant derivatives of the world function are given by (Synge, 1960)

$$\Omega_{i_P j_Q} = -g_{i_P j_Q} - h_{i_P j_Q} + O_2 \tag{4.3}$$

where $h_{i_P j_Q}$ is a 2-point tensor of the order O_1 [see (4.21)], and then by substituting these derivatives into the expression that can be obtained by taking second covariant derivatives with respect to P and Q in the solution for the geodesic triangle OPQ,

$$\Omega(PQ) = \Omega(OP) + \Omega(OQ) - \Omega_{i_O}(OP)\Omega^{i_O}(OQ) + \tfrac{1}{2}\phi(OPQ) \tag{4.4}$$

which is analogous to (3.7).

On the other hand, if we consider any geodesic square $PQRS$ in $\mathcal{N}$ and its geodesic diagonal Γ_{PR}, we have for the two triangles PQR and PSR

$$\Omega(PR)=\Omega(QP)+\Omega(QR)-\Omega_{i_Q}(QP)\Omega^{i_Q}(QR)+\tfrac{1}{2}\phi(PQR)$$
$$=\Omega(PS)+\Omega(SR)-\Omega_{i_S}(SP)\Omega^{i_S}(SR)+\tfrac{1}{2}\phi(PSR) \qquad (4.5)$$

so that, taking covariant derivatives in this expression with respect to P and R, we have

$$-\Omega_{i_P}^{j_Q}(QP)\Omega_{j_Q}^{l_R}(QR)+\tfrac{1}{2}\phi_{i_P}^{l_R}(PQR)=-\Omega_{i_P}^{m_S}(SP)\Omega_{m_S}^{l_R}(SR)+\tfrac{1}{2}\phi_{i_P}^{l_R}(PSR) \qquad (4.6)$$

and therefore, taking into account (4.3) again, we finally have

$$(g_{i_P}^{j_Q}+h_{i_P}^{j_Q})(g_{j_Q}^{l_R}+h_{j_Q}^{l_R})-(g_{i_P}^{m_S}+h_{i_P}^{m_S})(g_{m_S}^{l_R}+h_{m_S}^{l_R})=\tfrac{1}{2}[\phi_{i_P}^{l_R}(PQR)-\phi_{i_P}^{l_R}(PSR)] \qquad (4.7)$$

from which we deduce the following result: for a geodesic square $PQRS$ we have

$$g_{i_P}^{j_Q}g_{j_Q}^{l_R}-g_{i_P}^{m_S}g_{m_S}^{l_R}=g_{i_P}^{m_S}h_{m_S}^{l_R}+h_{i_P}^{m_S}g_{m_S}^{l_R}-g_{i_P}^{j_Q}h_{j_Q}^{l_R}-h_{i_P}^{j_Q}g_{j_Q}^{l_R}$$
$$+\tfrac{1}{2}\phi_{i_P}^{l_R}(PQR)-\tfrac{1}{2}\phi_{i_P}^{l_R}(PSR)+O_2 \qquad (4.8)$$

Now, with these two results we are in a condition of simplifying (3.20) by obtaining an expression equivalent to the second and third terms of its right-hand side, i.e., to

$$-[\chi_{i_A}(ABD)-\chi_{i_A}(ACD)]\lambda_{(a)}^{i_A}+[\chi_{i_D}(ABD)-\chi_{i_D}(ACD)]L_{(a)}^{(b)}\lambda_{(b)}^{i_D} \qquad (4.9)$$

in the first approximation with respect to the Riemann tensor.

In effect, if we take into account (3.13) and (2.8), from (4.9) we have

$$-[\Omega_{j_Bi_A}(BA)\Omega^{j_B}(BD)-\Omega_{l_Ci_A}(CA)\Omega^{l_C}(CD)]\lambda_{(a)}^{i_A}$$
$$+[\Omega_{j_B}(BA)\Omega_{k_D}^{j_B}(BD)-\Omega_{l_C}(CA)\Omega_{k_D}^{l_C}(CD)]\mu_{(a)}^{k_D} \qquad (4.10)$$

Next, taking into account (4.3), from (4.10) we have

$$[(g_{j_Bi_A}+h_{j_Bi_A})\Omega^{j_B}(BD)-(g_{l_Ci_A}+h_{l_Ci_A})\Omega^{l_C}(CD)]\lambda_{(a)}^{i_A}$$
$$-[\Omega_{j_B}(BA)(g_{k_D}^{j_B}+h_{k_D}^{j_B})-\Omega_{l_C}(CA)(g_{k_D}^{l_C}+h_{k_D}^{l_C}+h_{k_D}^{l_C})]\mu_{(a)}^{k_D}+O_2 \quad (4.11)$$

Then, taking into account (2.10), from (4.11) we have

$$g_{j_B}^{i_A}g_{k_D}^{j_B}[-\Omega^{k_D}(BD)\lambda_{(a)i_A}+\Omega_{i_A}(BA)\mu_{(a)}^{k_D}]-h_{j_Bi_A}g_{k_D}^{j_B}\Omega^{k_D}(BD)\lambda_{(a)}^{i_A}$$
$$+h_{k_D}^{j_B}g_{j_B}^{i_A}\Omega_{i_A}(BA)\mu_{(a)}^{k_D}+\Omega^{k_D}(CD)\mu_{(a)k_D}+h_{l_Ci_A}g_{k_D}^{l_C}\Omega^{k_D}(CD)\lambda_{(a)}^{i_A}$$
$$-\Omega^{i_A}(CA)\lambda_{(a)i_A}-h_{k_D}^{l_C}g_{l_C}^{i_A}\Omega_{i_A}(CA)\mu_{(a)}^{k_D}+O_2 \qquad (4.12)$$

Now, taking into account (4.2), from (4.12) we have

$$\{g^{l_C}_{k_D} g^{i_A}_{l_C} + g^{l_C}_{k_D} h^{i_A}_{l_C} + h^{l_C}_{k_D} g^{i_A}_{l_C} + \tfrac{1}{2}[\phi^{i_A}_{k_D}(ABD) - \phi^{i_A}_{k_D}(ACD)]\}$$
$$[-\Omega^{k_D}(BD)\lambda_{(\alpha)i_A} + \Omega_i(BA)\mu^{k_D}_{(\alpha)}] + \Omega^{k_D}(CD)\mu_{(\alpha)k_D} - \Omega^{i_A}(CA)\lambda_{(\alpha)i_A}$$
$$+ h_{l_C i_A} g^{l_C}_{k_D} \Omega^{k_D}(CD)\lambda^{i_A}_{(\alpha)} - h^{l_C}_{k_D} g^{i_A}_{l_C} \Omega_{i_A}(CA)\mu^{k_D}_{(\alpha)} + O_2 \tag{4.13}$$

and finally, taking into account (3.5), (3.6), and (3.15), from (4.13) we have

$$L^{(\beta)}_{(\alpha)} X_{(\beta)}(B)|_D - X_{(\alpha)}(B)|_A + X_{(\alpha)}(C)|_A + \Omega^{k_D}(CD)\mu_{(\alpha)k_D}$$
$$+ g^{l_C}_{k_D} h^{i_A}_{l_C}[\Omega^{k_D}(CD) - \Omega^{k_D}(BD)]\lambda_{(\alpha)i_A} + h^{l_C}_{k_D} g^{i_A}_{l_C}[\Omega_{i_A}(BA) - \Omega_{i_A}(CA)]\mu^{k_D}_{(\alpha)}$$
$$+ \tfrac{1}{2}[\phi^{i_A}_{k_D}(\mathrm{ABD}) - \phi^{i_A}_{k_D}(\mathrm{ACD})]$$
$$\times [-\Omega^{k_D}(\mathrm{BD})\lambda_{(\alpha)i_A} + \Omega_{i_A}(\mathrm{BA})\mu^{k_D}_{(\alpha)}] + \mathrm{O}_2 \tag{4.14}$$

Therefore, substituting (4.14) in (3.20), we have the following result: under the assumption of small curvature (O_1) (3.20), reduces to

$$X_{(\alpha)}(B)|_A - X_{(\alpha)}(C)|_A - X_{(\beta)}(B)|_d L^{(\beta)}_{(\alpha)}$$
$$= \Omega_{i_D}(\mathrm{CD}) L^{(4)}_{(\alpha)} \lambda^{\prime i_D}_{(4)}$$
$$+ \tfrac{1}{2} g^{l_C}_{k_D} h^{i_A}_{l_C}[\Omega^{k_D}(\mathrm{CD}) - \Omega^{k_D}(\mathrm{BD})]\lambda_{(\alpha)i_A} + \tfrac{1}{2} h^{l_C}_{k_D} g^{i_A}_{l_C}[\Omega_{i_A}(\mathrm{BA}) - \Omega_{i_A}(\mathrm{CA})]\mu^{k_D}_{(\alpha)}$$
$$+ \tfrac{1}{4}[\phi_{i_A}(\mathrm{ABD}) - \phi_{i_A}(\mathrm{ACD})]\lambda^{i_A}_{(\alpha)} - \tfrac{1}{4}[\phi_{k_D}(\mathrm{ABD}) - \phi_{k_D}(\mathrm{ACD})]\mu^{k_D}_{(\alpha)}$$
$$+ \tfrac{1}{4}[\phi^{i_A}_{k_D}(\mathrm{ABD}) - \phi^{i_A}_{k_D}(\mathrm{ACD})][-\Omega^{k_D}(\mathrm{BD})\lambda_{(\alpha)i_A} + \Omega_{i_A}(\mathrm{BA})\mu^{k_D}_{(\alpha)}] + \mathrm{O}_2 \tag{4.15}$$

where $\phi(ACD)$ is given by (Synge, 1960)

$$\phi(ACD) = \phi_0(ACD) + \phi_1(ACD) + \phi_2(ACD) + O_2$$
$$\phi_0(\mathrm{ACD}) = 3\mathrm{k}^3 \int_0^1 (1-\theta)^3 \, d\phi \int_{u_1}^{u_2} [(\mathrm{u}_2 - \mathrm{u})^2 + (\mathrm{u} - \mathrm{u}_1)^2]\{1122\} \, \mathrm{du}$$
$$\phi_1(\mathrm{ACD}) = 2\mathrm{k}^3 \int_0^1 \theta(1-\theta)^3 \, d\theta \int_{u_1}^{u_2} [2(\mathrm{u}_2 - \mathrm{u})^3\{11221\}$$
$$+ 3(u_2 - u)^2(u - u_1)\{11222\} + 3(u_2 - u)(u - u_1)^2\{22111\}$$
$$+ 2(u - u_1)^3\{22112\}] \, du \tag{4.16}$$

$$\phi_2(ACD)=\tfrac{1}{2}k^3\int_0^1 \theta^2(1-\theta)^3\,d\theta\int_{u_1}^{u_2}[(u_2-u)^4\{112211\}$$
$$+4(u_2-u)^3(u-u_1)\{112212\}$$
$$+3(u_2-u)^2(u-u_1)^2(\{112222\}+\{221111\})$$
$$+4(u_2-u)(u-u_1)^3\{221121\}+(u-u_1)^4\{221122\}]\,du$$
$$k^{-1}=u_2-u_1$$

θ in this expression was already described in (3.10) and u is an affine parameter running from a current point Q_1 of Γ_{AC} to the current point Q_2 of Γ_{AD} with the same value of θ as Q_1 has, along the geodesic $\Gamma_{Q_1Q_2}$ so that $u=u_1$ on AC and $u=u_2$ on AD. The three parts into which $\phi(ACD)$ is split involve, respectively, the Riemann tensor itself through the nonholonomic components of the symmetrized Riemann tensor in the symbol $\{1122\}$, its first-order covariant derivatives with respect to Q_1 and Q_2, denoted, respectively, by 1 and 2 in the fifth place of the 5-index symbols, and its second-order covariant derivatives appearing in the last two places of the 6-index symbols. Thus, for example,

$$\{1122\}=S_{(abcd)}[X^{(a)}(C)|_A][X^{(b)}(C)|_A[X^{(c)}(D)|_A][X^{(d)}(D)|_A] \tag{4.17}$$

and

$$\{11221\}=S_{(abcde)}[X^{(a)}(C)|_A][X^{(b)}(C)|_A][X^{(c)}(D)|_A]$$
$$\times[X^{(d)}(D)|_A][X^{(e)}(C)|_A] \tag{4.18}$$

where

$$S_{(abcd)}=S_{i_Aj_Ak_Al_A}\lambda^{i_A}_{(a)}\lambda^{j_A}_{(b)}\lambda^{k_A}_{(c)}\lambda^{l_A}_{(d)}$$
$$S_{(abcde)}=S_{i_Aj_Ak_Al_Am_A}\lambda^{i_A}_{(a)}\lambda^{j_A}_{(b)}\lambda^{k_A}_{(c)}\lambda^{l_A}_{(d)}\lambda^{m_A}_{(e)} \tag{4.19}$$

$S_{i_Aj_Ak_Am_A}$ being the symmetrized Riemann tensor

$$S_{ijkl}=-\tfrac{1}{3}(R_{ijkl}+R_{iljk}) \tag{4.20}$$

evaluated at A [a similar expression for $\phi(ABD)$ can be found in (5.5)].

Now, to evaluate $h^{i_A}_{l_C}$ and $h^{l_C}_{k_D}$ in (4.15) we need the general expression for any two points P and Q in $\mathscr{N}$. This can be found in Synge (1960): The two-point tensor $h_{i_Pj_Q}$ is given by the integral

$$h_{i_Pj_Q}=-\tfrac{3}{2}\int_0^1(1-\sigma)\sigma g_{i_Pa}g_{j_Qb}S^{abkl}U_kU_l\,d\sigma \tag{4.21}$$

taken along the geodesic Γ_{PQ} of equations $x^i = x^i(\sigma)$ with $U^i = dx^i/d\sigma$, and where σ is an affine parameter such that $x^{i_P} = x^i(0)$ and $x^{i_Q} = x^i(1)$. Therefore, according to (2.2), we have

$$\Omega_{i_P}(PQ) = -U_{i_P}, \qquad \Omega_{i_Q}(PQ) = U_{i_Q} \tag{4.22}$$

and, since the nonholonomic components $h_{(ab)}$ in an orthonormal basis $\lambda_{(c)}$ parallel transported along Γ_{PQ} are related to $h_{i_P j_Q}$ by

$$h_{i_P j_Q} = h_{(ab)} \lambda_{i_P}^{(a)} \lambda_{i_Q}^{(b)} \tag{4.23}$$

then from (4.21) and (4.22) we have

$$h_{(ab)} = -\tfrac{3}{2} U^{(c)} U^{(d)} \int_0^1 (1-\sigma)\sigma S_{(abcd)}\, d\sigma \tag{4.24}$$

Now, taking an affine parameter σ_1 for the geodesic Γ_{AC} with $\sigma_1 = 0$ at A and $\sigma_1 = 1$ at C, from (4.19), (4.23), and (4.24) we have

$$\begin{aligned} h_{i_A l_C} &= h_{(ab)} \lambda_{i_A}^{(a)} \mu_{l_C}^{(b)} \\ &= -\frac{3}{2}\left[\int_0^1 (1-\sigma_1)\sigma_1 S_{(abcd)}\, d\sigma_1\right] \Omega^{(c)}(AC)\Omega^{(d)}(AC)\lambda_{i_A}^{(a)}\mu_{l_C}^{(b)} \end{aligned} \tag{4.25}$$

and, since

$$\Omega^{(\alpha)}(AC) = -X^{(\alpha)}(C)|_A, \qquad \Omega^{(4)}(AC) = 0 \tag{4.26}$$

from (4.25) we have

$$h_{i_A l_C} = \underset{1}{K}_{(ab\gamma\delta)} [X^{(\gamma)}(C)|_A][X^{(\delta)}(C)|_A] \lambda_{i_A}^{(a)} \mu_{l_C}^{(b)} \tag{4.27}$$

where

$$\underset{1}{K}_{(ab\gamma\delta)} = -\tfrac{3}{2} \int_0^1 (1-\sigma_1)\sigma_1 S_{(ab\gamma\delta)}\, d\sigma_1 \tag{4.28}$$

On the other hand, taking an affine parameter σ_2 for Γ_{CD} so that $\sigma_2 = 0$ at C and $\sigma_2 = 1$ at D, we have

$$\begin{aligned} h_{l_C k_D} &= h_{(ab)} \lambda_{l_C}^{\prime(a)} \lambda_{k_D}^{\prime(b)} \\ &= -\frac{3}{2}\left[\int_0^1 (1-\sigma_2)\sigma_2 S_{(abcd)}\, d\sigma_2\right] \Omega^{(c)}(CD)\Omega^{(d)}(CD)\lambda_{l_C}^{\prime(a)}\lambda_{k_D}^{\prime(b)} \end{aligned} \tag{4.29}$$

and, since

$$\Omega_{(r)}(CD) = \Omega_{l_C}(CD)\lambda_{(r)}^{\prime l_C} = -(\Delta s')\lambda'_{(4)l_C}\lambda_{(r)}^{\prime l_C} = -(\Delta s')\eta_{(4r)} \tag{4.30}$$

then we have

$$h_{l_C k_D} = \underset{2}{K}_{(ab44)} (\Delta s')^2 \lambda'^{(a)}_{l_C} \lambda'^{(b)}_{k_D} \tag{4.31}$$

where

$$\underset{2}{K}_{(ab44)} = -\tfrac{3}{2} \int_0^1 (1-\sigma_2)\sigma_2 S_{(ab44)}\, d\sigma_2 \tag{4.32}$$

Therefore, taking into account (4.27), (4.31), and that $h^{i_A}_{l_C} = g^{i_A j_A} h_{j_A l_C}$, we have

$$\begin{aligned}
&\tfrac{1}{2} g^{l_C}_{k_D} h^{i_A}_{l_C} \Omega^{k_D}(CD) \lambda_{(a) i_A} \\
&\quad = \tfrac{1}{2} \mu^{(b)}_{k_D} \Omega^{k_D}(CD) [X^{(\gamma)}(C)|_A][X^{(\delta)}(C)|_A] \underset{1}{K}_{(ab\gamma\delta)} \\
&\quad = \tfrac{1}{2} L^{(b)}_{(d)} \lambda'^{(d)}_{k_D} \Omega^{k_D}(CD) [X^{(\gamma)}(C)|_A][X^{(\delta)}(C)|_A] \underset{1}{K}_{(ab\gamma\delta)} \\
&\quad = \tfrac{1}{2} L^{(b)}_{(4)}\, \Delta s' [X^{(\gamma)}(C)|_A][X^{(\delta)}(C)|_A] \underset{1}{K}_{(ab\gamma\delta)}
\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}
&-\tfrac{1}{2} g^{l_C}_{k_D} h^{i_A}_{l_C} \Omega^{k_D}(BD) \lambda_{(a) i_A} \\
&\quad = -\tfrac{1}{2} \mu^{(b)}_{k_D} \Omega^{k_D}(BD) [X^{(\gamma)}(C)|_A][X^{(\delta)}(C)|_A] \underset{1}{K}_{(ab\gamma\delta)} \\
&\quad = -\tfrac{1}{2} L^{(b)}_{(\eta)} \lambda'^{(\eta)}_{k_D} \Omega^{k_D}(BD) [X^{(\gamma)}(C)|_A][X^{(\delta)}(C)|_A] \underset{1}{K}_{(ab\gamma\delta)} \\
&\quad = \tfrac{1}{2} L^{(b)}_{(\mu)} [X^{(\mu)}(B)|_D][X^{(\gamma)}(C)|_A][X^{(\delta)}(C)|_A] \underset{1}{K}_{(ab\gamma\delta)}
\end{aligned} \tag{4.34}$$

In a similar way we have

$$\begin{aligned}
&\tfrac{1}{2} g^{i_A}_{l_C} h^{l_C}_{k_D} \Omega_{i_A}(BA) \mu^{k_D}_{(a)} \\
&\quad = \tfrac{1}{2} \mu'^{(a) i_A} \underset{2}{K}_{(ab44)} (\Delta s')^2 L^{(b)}_{(a)} \Omega_{i_A}(BA) \\
&\quad = -\tfrac{1}{2} L^{(b)}_{(a)} L^{-1(a)}_{(\beta)} (\Delta s')^2 [X^{(\beta)}(B)|_A] \underset{2}{K}_{(ab44)}
\end{aligned} \tag{4.35}$$

where

$$\mu'^{(a) i_A} = g^{i_A}_{l_C} \lambda'^{(a) l_C}$$

and

$$\begin{aligned}
&-\tfrac{1}{2} g^{i_A}_{l_C} h^{l_C}_{k_D} \Omega_{i_A}(CA) \mu^{k_D}_{(a)} \\
&\quad = -\tfrac{1}{2} \mu'^{(a) i_A} \underset{2}{K}_{(ab44)} (\Delta s')^2 L^{(b)}_{(a)} \Omega_{i_A}(CA) \\
&\quad = \tfrac{1}{2} L^{(b)}_{(a)} L^{-1(a)}_{(\beta)} (\Delta s')^2 [X^{(\beta)}(C)|_A] \underset{2}{K}_{(ab44)}
\end{aligned} \tag{4.36}$$

and therefore, using (4.33)–(4.36) in (4.15), we finally have, under the assumption of small curvature (O_1), that (4.15) reduces to

$$\begin{aligned} X_{(\alpha)}(B)|_A - X_{(\alpha)}(C)|_A - X_{(\beta)}(B)|_D L^{(\beta)}_{(\alpha)} \\ = L^{(4)}_{(\alpha)}\Delta s' \\ + \tfrac{1}{2}\underset{1}{K}_{(ab\gamma\delta)}[X^{(\gamma)}(C)|_A][X^{(\delta)}(C)|_A][L^{(b)}_{(4)}\Delta s' + L^{(b)}_{(\mu)}X^{(\mu)}(B)|_D] \\ + \tfrac{1}{2}\underset{2}{K}_{(ab44)}L^{(b)}_{(\alpha)}L^{-1(\alpha)}_{(\beta)}(\Delta s')^2[X^{(\beta)}(C)|_A - X^{(\beta)}(B)|_A] \\ + \tfrac{1}{4}[\phi_{i_A}(ABD) - \phi_{i_A}(ACD)]\lambda^{i_A}_{(\alpha)} - \tfrac{1}{4}[\phi_{k_D}(ABD) - \phi_{k_D}(ACD)]\mu^{k_D}_{(\alpha)} \\ + \tfrac{1}{4}[\phi^{i_A}_{k_D}(ABD) - \phi^{i_A}_{k_D}(ACD)][-\Omega^{k_D}(BD)\lambda_{(\alpha)i_A} + \Omega_{i_A}(BA)\mu^{k_D}_{(\alpha)}] + O_2 \end{aligned} \tag{4.37}$$

since

$$\Omega_{i_D}(CD)L^{(4)}_{(\alpha)}\lambda'^{i_D}_{(4)} = L^{(4)}_{(\alpha)}\Delta s'$$

The interpretation of (4.37) is easier than that of (4.15) because, since $X_{(\beta)}(B)|_D = X^{(\beta)}(B)|_D$ and $X^{(4)}(D)|_C = \Delta s'$, then the third term of the left-hand side of (4.37) and the first term of its right-hand side can be put together in the expression

$$X^{(\beta)}(B)|_D L^{(\alpha)}_{(\beta)} + X^{(4)}(D)|_C L^{(\alpha)}_{(4)} \tag{4.38}$$

so that these two terms appear as the Fermi coordinates,

$$(x^{(\beta)}(B),\ x^{(4)}(B) + \text{Cns.})$$

of B with respect to the baseline Γ, considering a rotating or nonrotating reference frame, depending respectively, on whether the Lorentz matrix is taken into account or not. On the other hand, when $b = \beta$ one can see inside the brackets of the second term of the right-hand side of (4.37) the same expression (4.38), so that this whole second term appears as a correction due to the Riemann tensor of the two former terms cited previously. Finally, it can be seen that the third term of the right-hand side is a correction of the first and second terms of the left-hand side due also to the Riemann tensor. Now, it is better to defer an interpretation of the remaining three terms of the right-hand side of (4.37) until after applying the hypothesis of quasiparallelism (this will be made in the next section). Obviously, when the Riemann tensor vanishes, (4.37) reduces to the formula (3.30), which corresponds to the flat-space-time particular case.

5. APPROXIMATION BY QUASIPARALLELISM

Let us suppose that U^{l_C} is a tangent vector to the geodesic Γ at C and that U^{i_A} is a tangent vector of Π at A. We introduce here the following hypothesis: the nonholonomic components in the tetrad $\lambda^{i_A}_{(a)}$ of the image of U^{l_C} at A by the parallel propagator $g^{i_A}_{l_C}$ along the geodesic Γ_{AC}, $g^{i_A}_{l_C}U^{l_C}$, differ from the corresponding nonholonomic components of U^{i_A} in the same tetrad in quantities of the order O_1, i.e.,

$$g^{i_A}_{l_C}U^{l_C}\lambda_{(a)i_A} = U^{i_A}\lambda_{(a)i_A} + O_1 \tag{5.1}$$

Now, with this hypothesis of quasiparallelism between the world lines Π and Γ it can be shown that $\Omega(CD)$ is of the order O_1. In fact, considering the triangles ABC, ABD, and BCD, we have

$$\Omega(BC) = \Omega(AB) + \Omega(AC) - \Omega_{i_A}(AB)\Omega^{i_A}(AC) + \tfrac{1}{2}\phi(ABC) \tag{5.2}$$

$$\Omega(BD) = \Omega(AD) + \Omega(AB) - \Omega_{i_A}(AD)\Omega^{i_A}(AB) + \tfrac{1}{2}\phi(ABD) \tag{5.3}$$

$$\Omega(BC) = \Omega(BD) + \Omega(DC) + \tfrac{1}{2}\phi(BCD) \tag{5.4}$$

where $\phi(ABC)$, $\phi(ABD)$, and $\phi(BCD)$ are O_1 quantities which have similar values to $\phi(ACD)$ given in (4.16). For example,

$$\begin{aligned}
\phi(ABD) &= \phi_0(ABD) + \phi_1(ABD) + \phi_2(ABD) + O_2 \\
\phi_0(ABD) &= 3\bar{k}^3 \int_0^1 (1-\zeta)^3\, d\zeta \int_{\bar{u}_1}^{\bar{u}_2} [(\bar{u}_2-\bar{u})^2 + (\bar{u}-\bar{u}_1)^2]\{1122\}\, d\bar{u} \\
\phi_1(ABD) &= 2\bar{k}^3 \int_0^1 \zeta(1-\zeta)^3\, d\zeta \int_{\bar{u}_1}^{\bar{u}_2} [2(\bar{u}_2-\bar{u})^3\{11221\} \\
&\quad + 3(\bar{u}_2-\bar{u})^2(\bar{u}-\bar{u}_1)\{11222\} \\
&\quad + 3(\bar{u}_2-\bar{u})(\bar{u}-\bar{u}_1)^2\{22111\} \\
&\quad + 2(\bar{u}-\bar{u}_1)^3\{22112\}]\, d\bar{u} \\
\phi_2(ABD) &= \tfrac{1}{2}\bar{k}^3 \int_0^1 \zeta^2(1-\zeta)^3\, d\zeta \int_{\bar{u}_1}^{\bar{u}_2} [(\bar{u}_2-\bar{u})^4\{112211\} \\
&\quad + 4(\bar{u}_2-\bar{u})^3(\bar{u}-\bar{u}_1)\{112212\} \\
&\quad + 3(\bar{u}_2-\bar{u})^2(\bar{u}-\bar{u}_1)^2(\{112222\} + \{221111\}) \\
&\quad + 4(\bar{u}_2-\bar{u})(\bar{u}-\bar{u}_1)^3\{221121\} + (\bar{u}-\bar{u}_1)^4\{221122\}]\, d\bar{u} \\
\bar{k}^{-1} &= \bar{u}_2 - \bar{u}_1
\end{aligned} \tag{5.5}$$

were ζ was already described in (3.8) and $\bar{u}$ is an affine parameter running now from the geodesic Γ_{BD}, where $\bar{u}=\bar{u}_1$, to the geodesic $\Gamma_{B\delta}$, where $\bar{u}=\bar{u}_2$.

Then, taking $\Omega(AB)$ and $\Omega(BC)$ from (5.2) and (5.3) to (5.4), we have

$$\Omega(CD)=\Omega(AC)-\Omega(AD)+\Omega_{i_A}(AD)\Omega^{i_A}(AB)-\Omega_{i_A}(AB)\Omega^{i_A}(AC)+O_1 \tag{5.6}$$

and, since from (5.1) we have

$$\Omega(AD)=\Omega(AC)+\Omega(CD)+O_1 \tag{5.7}$$

then from (5.6) we have

$$2\Omega(CD)=\Omega^{i_A}(AB)[\Omega_{i_A}(AD)-\Omega_{i_A}(AC)]+O_1 \tag{5.8}$$

On the other hand, taking covariant derivatives of (3.9) with respect to the coordinates of A, we have

$$\Omega_{i_A}(AD)=\Omega_{i_A}(AC)-\Omega_{l_C i_A}(AC)\Omega^{l_C}(CD)+O_1 \tag{5.9}$$

which with the help of (4.3) can also be written

$$\Omega_{i_A}(AD)-\Omega_{i_A}(AC)=g_{l_C i_A}\Omega^{l_C}(CD)+O_1 \tag{5.10}$$

so that, by substituting (5.10) in (5.8), we have

$$2\Omega(CD)=\Omega^{i_A}(AB)g_{l_C i_A}\Omega^{l_C}(CD)+O_1 \tag{5.11}$$

But, since again by (5.1) we have

$$\Omega^{i_A}(AB)g_{l_C i_A}\Omega^{l_C}(CD)=O_1 \tag{5.12}$$

then from (5.11) we finally have

$$\Omega(CD)=O_1 \quad \text{QED} \tag{5.13}$$

Note that (5.7) and (5.12) come essentially from the hypothesis of quasiparallelism, because under this hypothesis the Lorentz matrix $L^{(a)}_{(b)}$ given by (3.15), which according to (3.14) can be written

$$L^{(a)}_{(b)}=\lambda'^{(a)}_{l_C}\mu^{l_C}_{(b)}=\lambda'^{(a)}_{l_C}\, g^{l_C}_{i_A}\lambda^{i_A}_{(b)} \tag{5.14}$$

satisfies

$$L^{(a)}_{(4)}=\delta^{(a)}_{(4)}+l^{(a)}_{(4)} \tag{5.15}$$

with $l^{(a)}_{(4)}=O_1$. Then, (5.1) together with (5.15) tells what this hypothesis means: that the relative velocity between the observers Π and Γ is small, compared with the velocity of light.

Now, with this result it is possible to simplify the expression (4.16) for $\phi(ACD)$ because, if we take into account (5.10) and (5.13), we have

$$\begin{aligned}\Omega_{i_A}(AD)\lambda^{i_A}_{(a)} &= \Omega_{i_A}(AC)\lambda^{i_A}_{(a)} + O_1\\ \Omega_{i_A}(AD)\lambda^{i_A}_{(4)} &= \Omega_{i_A}(AC)\lambda^{i_A}_{(4)} + O_1 = O_1\end{aligned} \tag{5.16}$$

and, therefore, by letting u_2 tend to u_1 so that the triangle ACD becomes very thin, we see that $\phi_1(ACD)$ and $\phi_2(ACD)$ become O_2, while $\phi_0(ACD)$ reduces to

$$\phi_0(ACD) = 2S_{(abcd)}[X^{(a)}(C)|_A][X^{(b)}(C)|_A][X^{(c)}(D)|_A][X^{(d)}(D)|_A] \tag{5.17}$$

which, in turn, on account of (5.16), reduces to

$$\begin{aligned}\phi_0(ACD) &= 2S_{(abcd)}[\Omega^{i_A}(AC)\lambda^{(a)}_{i_A}\Omega^{j_A}(AC)\lambda^{(b)}_{j_A}\Omega^{k_A}(AC)\lambda^{(c)}_{k_A}\Omega^{l_A}(AC)\lambda^{(d)}_{l_A} + O_1]\\ &= 2S_{(abcd)}\{[X^{(a)}(C)|_A][X^{(b)}(C)|_A][X^{(c)}(C)|_A][X^{(d)}(C)|_A] + O_1\}\end{aligned} \tag{5.18}$$

Now, this expression is O_2 because

$$S_{(abcd)}[X^{(a)}(C)|_A][X^{(b)}(C)|_A][X^{(c)}(C)|_A][X^{(d)}(C)|_A]$$

vanishes due to the skew-symmetry of the Riemann tensor. Therefore, taking into account (5.18) and that $(\Delta s')^2 \cong \Omega(CD) = O_1$, we finally have the following result: under the assumption of quasiparallelism (4.37) reduces to

$$\begin{aligned}&X_{(\alpha)}(B)|_A - X_{(\alpha)}C)|_A - X_{(\beta)}(B)|_D L^{(\beta)}_{(\alpha)}\\ &\quad = [\underset{1}{1}^{(4)}_{(\alpha)} + \tfrac{1}{2}\underset{1}{\mathrm{K}}_{(\alpha4\gamma\delta)}\{[\mathrm{X}^{(\gamma)}(\mathrm{C})|_A][\mathrm{X}^{(\delta)}(\mathrm{C})|_A]\}\ \Delta s'\\ &\quad\quad + \tfrac{1}{2}\underset{1}{\mathrm{K}}_{(\alpha\beta\gamma\delta)}[\mathrm{X}^{(\gamma)}(\mathrm{C})|_A][\mathrm{X}^{(\delta)}(\mathrm{C})|_A]\mathrm{L}^{(\beta)}_{(\mu)}[\mathrm{X}^{(\mu)}(\mathrm{B})|_D]\\ &\quad\quad + \tfrac{1}{4}[\phi_{i_A}(\mathrm{ABD})\lambda^{i_A}_{(\alpha)} - \phi_{k_D}(\mathrm{ABD})\mathrm{L}^{(\beta)}_{(\alpha)}\lambda'^{k_D}_{(\beta)}]\\ &\quad\quad + \tfrac{1}{4}[\phi^{i_A}_{k_D}(\mathrm{ABD})][-\Omega^{k_D}(\mathrm{BD})\lambda_{(\alpha)i_A} + \Omega_{i_A}(\mathrm{BA})\mathrm{L}^{(\beta)}_{(\alpha)}\lambda'^{k_D}_{(\beta)}] + \mathrm{O}_2\end{aligned} \tag{5.19}$$

where $\phi(ABD)$ is given by (5.5).

According to (3.5) and (3.6), we see in the two first terms of the left-hand side of (5.19) the Fermi coordinates of B and C, $x_{(\alpha)}(B)$ and $x_{(\alpha)}(C)$, with respect to the nonrotating proper reference frame $\lambda^i_{(a)}$ chosen for Π, and in the third term the Fermi coordinates of B, $x'_{(\beta)}(B)$ and $L^{(\beta)}_{(\alpha)}x'_{(\beta)}$, with respect, respectively, to the nonrotating proper reference frame $\lambda'^i_{(a)}$ and the rotating proper reference frame $\mu^i_{(a)}$ related to $\lambda'^i_{(a)}$ by (3.15), both attached to Γ. The meaning of the first and second terms of the right side was already discussed at the end of the previous section. Nevertheless, it is to be noted that, in passing from (4.37) to (5.19), only the O_1 factors $\underset{1}{l}^{(4)}_{(\alpha)}$, $\underset{1}{K}_{(\alpha4\gamma\delta)}$, and $\underset{1}{K}_{(\alpha\beta\gamma\delta)}$ have survived. As can be seen in (4.28) and (4.32), these two curvature factors $\underset{1}{K}_{(\alpha4\gamma\delta)}$ and $\underset{1}{K}_{(\alpha\beta\gamma\delta)}$, together with the aberration factors $\phi_{i_A}(ABD)$, $\phi_{k_D}(ABD)$, and $\phi^{i_A}_{k_D}(ABD)$ (which we can now simply write ϕ_{i_A}, ϕ_{k_D}, and $\phi^{i_A}_{k_D}$) contain the information, through its Riemann tensor of the particular space-time chosen for the transformation. (Note that this tensor is to be evaluated on the nongeodesic baseline.) Of course, although the contribution of these two terms will not be null in general, it is known that the whole first term may vanish in some particular space-times. Something similar happens with the two last terms of the right-hand side: in general, it is not possible to cause to disappear in (5.19) any one of the three terms ϕ_0, ϕ_1, and ϕ_2 into which $\phi(ABC)$ is split to obtain a new transformation general enough to be valid in any realistic situation. For example, we can make $\phi(ABD)$ reduce to ϕ_0 by assuming $\Omega(AD)$ small compared to $\Omega(BA)$; but by doing this, since ϕ_1 and ϕ_2 contain inner products of the symmetrized Riemann tensor with the Fermi coordinates of the event observed, B, with respect to the two baselines Π and Γ through symbols like $\{11222\}$, then their contribution to the velocity of recession between the two baselines contained in $\phi^{i_A}_{k_D}$ would be neglected, so that the new transformation becomes imprecise to observe, in this case, near objects. Therefore, although ϕ_{i_A}, ϕ_{k_D}, and $\phi^{i_A}_{k_D}$ can be obtained in general by using the expanding series solution of the geodesic equation for Γ_{AD}, the evaluation of ϕ_0, ϕ_1, and ϕ_2 must be deferred until a dynamical model has been selected. Writing, then, the last term in (5.19) in the form

$$\tfrac{1}{4}[\phi^{i_A}_{k_D}][x'_{(\beta)}(B)\lambda'^{(\beta)k_D}\lambda_{(\alpha)i_A} - x_{(\gamma)}(B)\lambda^{(\gamma)}_{i_A}L^{(\beta)}_{(\alpha)}\lambda'^{k_D}_{(\beta)}] \tag{5.20}$$

to better show this circumstance, we finally have the following transformation:

$$\begin{aligned} x_{(\alpha)}(B) - x_{(\alpha)}(C) - x'_{(\beta)}(B)L^{(\beta)}_{(\alpha)} &= [1^{(4)}_{(\alpha)} + \underset{1}{\bar{K}}_{(\alpha 4)}]\Delta s' \\ &+ \underset{1}{\bar{K}}_{(\alpha\beta)}L^{(\beta)}_{(\mu)}x'^{(\mu)}(B) \\ &+ \tfrac{1}{4}[\phi_{i_A}\lambda^{i_A}_{(\alpha)} - \phi_{k_D}L^{(\beta)}_{(\alpha)}\lambda'^{k_D}_{(\beta)}] \\ &+ \tfrac{1}{4}[\phi^{i_A}_{k_D}][x'_{(\beta)}(B)\lambda'^{k_D}_{(\beta)}\lambda_{(\alpha)i_A} \\ &- x_{(\gamma)}(B)\lambda^{(\gamma)}_{i_A}L^{(\beta)}_{(\alpha)}\lambda'^{k_D}_{(\beta)}] + O_2 \end{aligned} \tag{5.21}$$

where $\underset{1}{\bar{K}}_{(\alpha 4)}$ and $\underset{1}{\bar{K}}_{(\alpha\beta)}$, which are given by

$$\begin{aligned} \underset{1}{\bar{K}}_{(\alpha 4)} &= \tfrac{1}{2}\underset{1}{K}_{(\alpha 4\gamma\delta)}x^{(\gamma)}(C)x^{(\delta)}(C) \\ \underset{1}{\bar{K}}_{(\alpha\beta)} &= \tfrac{1}{2}\underset{1}{K}_{(\alpha\beta\gamma\delta)}x^{(\gamma)}(C)x^{(\delta)}(C) \end{aligned} \tag{5.22}$$

have been introduced to show the final transformation in a more compact form. This transformation is now ready to input both the value of the Riemann tensor on the accelerated observer and the behavior of the two reference frames as characteristics derived from the particular selection of the observers and from the dynamical model to be chosen.

REFERENCES

Brumberg, V. A. (1979). *Celestial Mechanics*, **20**, 329.

Fukushima, T., Fujimoto, H., Kinoshita, H., and Aoki, S. (1986). In *Proceedings of the 114th Symposium of the IAU*, J. Kovalenski and V. A. Brumberg, eds., D. Reidel, Dordrecht.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, W. H. Freeman, San Francisco.

Synge, J. L. (1960). *Relativity: The General Theory*, North-Holland, Amsterdam.